§2.8 Subspaces of $\mathbb{R}^n$

<u>Defn</u>: A subspace of $\mathbb{R}^n$ is a subset $H \subseteq \mathbb{R}^n$ satisfying the following:

1) The zero vector, 0 is in H.

2) For all v and w in H, $v+w$ is also in H.

3) For any v in H and any scalar c, $c \cdot v$ is in H.

(2) and (3) are often rephrased by saying H is "closed under addition and scalar multiplication".

What kind of subspaces have we seen?

<u>Remark</u>: If $v_1, \dots, v_\ell$ are vectors of $\mathbb{R}^n$, then $\text{span}\{v_1, \dots, v_\ell\}$ is a subspace of $\mathbb{R}^n$

<u>Proof</u>

1) $0 = 0v_1 + \dots + 0v_\ell$ so 0 is in $\text{span}\{v_1, \dots, v_\ell\}$

2) If w and v are in $\text{span}\{v_1, \dots, v_\ell\}$ we can write

$w = c_1 v_1 + \dots + c_\ell v_\ell$ and $v = d_1 v_1 + \dots + d_\ell v_\ell$

Then $v + w = (c_1 + d_1)v_1 + \dots + (c_\ell + d_\ell)v_\ell$

so $v+w$ is in $\text{span}\{v_1, \dots, v_\ell\}$

3) If w in span$\{v_1, \dots, v_\ell\}$ we can write $w = c_1 v_1 + \dots + c_\ell v_\ell$. Now if c is any scalar $c \cdot w = (c \cdot c_1) v_1 + \dots + (c \cdot c_\ell) v_\ell$ so $c \cdot w$ is in span$\{v_1, \dots, v_\ell\}$.

With this, we get nice geometric interpretations of some subspaces:

- If v is in $\mathbb{R}^n$, span$\{v\}$ is a line passing through the origin
- If v and w are linearly independent vectors in $\mathbb{R}^n$, the span$\{v, w\}$ is a plane passing through the origin

Remark: Lines and planes that don't pass through the origin are not subspaces! They don't contain 0!

As a matter of terminology we might call span$\{v_1, \dots, v_\ell\}$ as the subspace spanned by $v_1, \dots, v_\ell$.

Defn: Let A be an $m \times n$ matrix. The column space of A, $\text{Col}(A)$ is the set of all linear combinations of the columns of A.

In other words, $\text{Col}(A)$ is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

- Recall that we say the columns of A span $\mathbb{R}^m$ if $Ax = b$ has a solution for all b in $\mathbb{R}^m$ (equivalently if A has a pivot in each row!)

- In this case $\text{Col}(A) = \mathbb{R}^m$

- Just as before, in order to determine if b is in $\text{Col}(A)$ we just need to determine if $Ax = b$ has a solution (see §1.4)

Defn: Let A be an $m \times n$ matrix. The null space of A, $\text{Nul}(A)$ is the set of all solutions to the homogeneous matrix equation

$$Ax = 0$$

$\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

But what if $Ax = 0$ only has the trivial solution?

That's ok! Notice that the one element set $\{0\}$ (zero vector of $\mathbb{R}^n$) is a subspace (of $\mathbb{R}^n$). Check this!

We usually refer to $\{0\}$ as the <u>zero subspace</u> (of $\mathbb{R}^n$) and notate it by 0 (like everything else, just to make it confusing)

Notice that if we consider $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the map given by $T(x) = Ax$, then T is one-to-one if and only if $Nul(A) = 0$.

<u>Defn</u>: Let H be a subspace of $\mathbb{R}^n$. A <u>basis</u> of H is a linearly independent set $\{v_1, \dots, v_\ell\}$ that spans H, i.e. $H = \text{span}\{v_1, \dots, v_\ell\}$

<u>Example</u>: Let $e_1, \dots, e_n$ denote the columns of I_n

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots\dots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$\{e_1, \dots, e_n\}$ is a basis for $\mathbb{R}^n$. Check this!

Moreover, we call this the <u>standard basis</u> for $\mathbb{R}^n$.

Example

Find a basis for Nul(A) if $A = \begin{bmatrix} 1 & 2 & 3 & 6 & 4 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 1 & 4 & 0 & 4 \end{bmatrix}$

solution:

$$[A \mid 0] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 6 & 4 & 0 \\ 0 & 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 6 & 4 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -5 & 6 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

pivots

$$\begin{cases} x_1 - 5x_3 + 6x_4 = 0 \\ x_2 + 4x_3 = 0 \\ x_5 = 0 \end{cases} \implies \begin{cases} x_1 = 5x_3 - 6x_4 \\ x_2 = -4x_3 \\ x_3 = \text{free} \\ x_4 = \text{free} \\ x_5 = 0 \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_3 - 6x_4 \\ -4x_3 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{Nul}(A) = \text{solution set of } Ax = 0 = \text{span}\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Thus $\left\{ \begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for Nul(A).

Remark: We've really only shown this is a spanning set for Nul(A). However, since we got it by reducing A to reduced echelon form, it is also linearly independent. Why is this? Think about it.

Example

Again, let $A = \begin{bmatrix} 1 & 2 & 3 & 6 & 4 \\ 0 & 1 & 4 & 0 & 2 \\ 0 & 1 & 4 & 0 & 4 \end{bmatrix}$. Find a basis for Col(A).

Solution:

$$A \xrightarrow[\text{reduction}]{\text{same row}} \begin{bmatrix} 1 & 0 & -5 & 6 & 0 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot columns

Notice the pivot columns in the reduced echelon form are part of the standard basis of $\mathbb{R}^3$ (all of it in this case). Makes sense since Col(A) is a subspace of $\mathbb{R}^3$.

This tells us the pivot columns of A form a basis of Col A. Thus

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \right\}$$ is a basis for Col(A).

Warning!

The pivot columns of a matrix A form a basis of Col(A). The reduced echelon form only tells you which columns are pivot columns. The columns of the reduced echelon are not in Col(A) in general so we really need to use the pivot columns of A.

Example (of the warning)

Find a basis for $Col(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 5 & 10 \end{bmatrix}$

$$A \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right\}$ is a basis for $Col(A)$.

Notice the pivot columns of the reduce echelon form $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ won't work!

This is easy to see as $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ is in $Col(A)$ and every vector in $\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ has 0 as a 3^{rd} entry so $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ isn't in there.